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Fuzzy control of a class of multivariable nonlinear systems subject to parameter uncertainties: model reference approach

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Abstract

This paper presents the fuzzy control of a class of multivariable nonlinear systems subject to parameter uncertainties. The nonlinear plant tackled in this paper is an n th-order nonlinear system with n inputs. If the input matrix \mathbf{B} inside the fuzzy plant model is invertible, a fuzzy controller can be designed such that the states of the closed-loop system will follow those of a user-defined stable reference model despite the presence of parameter uncertainties. A numerical example will be given to show the design procedures and the merits of the proposed fuzzy controller. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Fuzzy control; Fuzzy plant model; Model reference; Multivariable nonlinear system; Parameter uncertainties; Stability

1. Introduction

Fuzzy control is one of the useful control techniques for uncertain and ill-defined nonlinear systems. Control actions of the fuzzy controller are described by some linguistic rules. This property makes the control algorithm to be understood easily. The early design of fuzzy controllers is heuristic. It

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incorporates the experience or knowledge of the designer into the rules of the fuzzy controller, which is fine tuned based on trial and error. A fuzzy controller implemented by neural network was proposed in [3,4]. Through the use of tuning methods, fuzzy rules can be generated automatically. These methodologies make the design simple; however, the design does not guarantee the system stability, robustness and good performance.

To facilitate the formal analysis of fuzzy control systems and the design of fuzzy controllers, Takagi–Sugeno (TS) fuzzy plant model was proposed in [1,13]. The TS model represents a nonlinear system as a weighted sum of some linear systems. Based on this structure, fuzzy controllers comprising a number of sub-controllers were proposed. State feedback controllers were proposed as the sub-controllers in [2,5–7,17,18]. The closed-loop system is guaranteed to be asymptotically stable if there exists a common solution for a number of linear matrix inequalities (LMI). Other stability conditions can be found in [10,12,13]. The LMI-based design of fuzzy controllers can also be found in [14–16].

The problem will become more complex if the fuzzy plant models are subject to parameter uncertainties. When \mathbf{A}_i and \mathbf{B}_i , which are the systems matrices of the fuzzy plant model, are subject to parameter uncertainties, e.g., $\Delta\mathbf{A}_i$ and $\Delta\mathbf{B}_i$, robustness analysis can be found in the literature. In [19,20], only the case that \mathbf{A}_i contains parameter uncertainty $\Delta\mathbf{A}_i$ was considered. Lyapunov stability theory was employed to find stability conditions and ranges of the elements of $\Delta\mathbf{A}_i$ such that the nonlinear plant connected with a fuzzy state feedback controller is stable. In [5,17,18], both \mathbf{A}_i and \mathbf{B}_i were considered to have parameter uncertainties. Stability conditions had been derived based on H_∞ theory [5] and by estimating the matrix measures of the system matrices of the fuzzy plant model [17,18]. It can be seen in [5,17–20] that their prime objective and the analysis results were on the system stability; the system performance is not considered at all. In [11], a sliding mode controller was proposed as the sub-controller. To compensate the effect caused by the parameter uncertainties to the system stability, a high gain switching component was used. Adaptive fuzzy controllers were also reported in [8,9] to deal with systems with unknown parameters. The value of each unknown parameter was estimated by the adaptive fuzzy controller. However, by introducing adaptivity to the fuzzy controller, the structure of the controller becomes more complex.

As mentioned above, some fuzzy state-feedback controllers guarantee only the system stability. Although stability is one essential aspect to be considered, other aspects including the system performance, are also important. In order to have a simple fuzzy state feedback controller which not only guarantees the system stability but also gives good robustness property and system performance, a design methodology will be proposed in this paper. Under the proposed design methodology, the system states of the closed-loop system will follow those of a stable user-defined reference model. A class of multivariable nonlinear system, an n th-order- n -input nonlinear system subject to parameter

uncertainties, will be considered in this paper. This system will be represented by a fuzzy plant model. The input matrix \mathbf{B} inside the fuzzy plant model is required to be invertible. This class of systems can be found in the real world. For instance, $\theta - r$ manipulators [21], two-wheeled mobile robots [22] and two-link robot arms [23] are examples of this class of systems. At the end of this paper, a numerical example will be given to show the design procedure and the merits of the proposed fuzzy controller.

2. Reference model, fuzzy plant model and fuzzy controller

An n th-order- n -input nonlinear plant subject to parameter uncertainties will be considered. This plant is represented by a fuzzy plant model that expresses the plant as a weighted sum of some linear systems with parameter uncertainty information. A fuzzy controller is to be designed to close the feedback loop such that the system states of the closed-loop system will follow those of a stable reference model.

2.1. Reference model

A reference model is a stable linear system given by,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t), \quad (1)$$

where $\mathbf{H}_m \in \mathfrak{R}^{n \times n}$ is a constant stable system matrix, $\mathbf{B}_m \in \mathfrak{R}^{n \times n}$ a constant input matrix, $\hat{\mathbf{x}} \in \mathfrak{R}^{n \times 1}$ the system state vector of this reference model and $\mathbf{r}(t) \in \mathfrak{R}^{n \times 1}$ is the bounded reference input.

2.2. Fuzzy plant model with parameter uncertainty information

Let p be the number of fuzzy rules describing the uncertain nonlinear plant. The i th rule is of the following form,

$$\begin{aligned} \text{Rule } i : & \text{ IF } x_1(t) \text{ is } M_1^i \text{ and } \dots \text{ and } x_n(t) \text{ is } M_n^i \\ & \text{ THEN } \dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t), \end{aligned} \quad (2)$$

where M_k^i is a fuzzy set of rule i corresponding to the state $x_k(t)$, $k = 1, 2, \dots, n$, $i = 1, 2, \dots, p$; $\mathbf{A}_i \in \mathfrak{R}^{n \times n}$ and $\mathbf{B}_i \in \mathfrak{R}^{n \times n}$ are the known system and input matrices, respectively; $\Delta\mathbf{A}_i \in \mathfrak{R}^{n \times n}$ is the parameter uncertainties of \mathbf{A}_i within known ranges; $\mathbf{x}(t) \in \mathfrak{R}^{n \times 1}$ is the system state vector and $\mathbf{u}(t) \in \mathfrak{R}^{n \times 1}$ is the input vector. The plant dynamics is described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))[(\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)], \quad (3)$$

where

$$\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0, 1] \quad \text{for all } i \quad (4)$$

is a known nonlinear function of $\mathbf{x}(t)$ and

$$w_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}(x_1(t)) \times \mu_{M_2^i}(x_2(t)) \times \cdots \times \mu_{M_n^i}(x_n(t))}{\sum_{j=1}^p (\mu_{M_1^j}(x_1(t)) \times \mu_{M_2^j}(x_2(t)) \times \cdots \times \mu_{M_n^j}(x_n(t)))}, \quad (5)$$

$\mu_{M_k^i}(x_k(t))$ is the grade of membership of the fuzzy set M_k^i .

2.3. Fuzzy controller

A fuzzy controller having p fuzzy rules is to be designed for the plant. The j th rule of the fuzzy controller is of the following format:

$$\text{Rule } j: \text{ IF } x_1(t) \text{ is } M_1^i \text{ and } \dots \text{ and } x_n(t) \text{ is } M_n^i \text{ THEN } \mathbf{u}(t) = \mathbf{u}_j(t), \quad (6)$$

where $\mathbf{u}_j(t) \in \mathfrak{R}^{n \times 1}$, $j = 1, 2, \dots, p$, is the output of the j th rule controller that will be defined in the next section. The global output of the fuzzy controller is given by

$$\mathbf{u}(t) = \sum_{j=1}^p w_j(\mathbf{x}(t)) \mathbf{u}_j(t). \quad (7)$$

3. Design of the fuzzy controller

In this section, we describe the design of the fuzzy controller, i.e., $\mathbf{u}_j(t)$ for $j = 1, 2, \dots, p$, such that the closed-loop system behaves like the stable reference model. From (3), (4) and (7), writing $w_i(\mathbf{x}(t))$ as w_i , we have,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^p w_i \left[(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B}_i \sum_{j=1}^p w_j \mathbf{u}_j(t) \right] \\ &= \left(\sum_{i=1}^p w_i (\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) \right) + \left(\sum_{i=1}^p w_i \mathbf{B}_i \right) \left(\sum_{j=1}^p w_j \mathbf{u}_j(t) \right) \\ &= \sum_{i=1}^p w_i (\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \sum_{j=1}^p w_j \mathbf{u}_j(t) \\ &= \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t)], \end{aligned} \quad (8)$$

where,

$$\mathbf{B} = \sum_{i=1}^p w_i \mathbf{B}_i. \quad (9)$$

Note that \mathbf{B} is a known function of \mathbf{x} . From (1) and (8) and the property that

$$\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1,$$

let,

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t)] - \mathbf{H}_m \hat{\mathbf{x}}(t) - \mathbf{B}_m \mathbf{r}(t) \\ &= \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t)] - \sum_{i=1}^p w_i (\mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t)) \\ &= \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t) - \mathbf{H}_m \hat{\mathbf{x}}(t) - \mathbf{B}_m \mathbf{r}(t)], \end{aligned} \quad (10)$$

where $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is an error vector. We define the following Lyapunov function,

$$V = \frac{1}{2} \mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t), \quad (11)$$

where $(\cdot)^T$ denotes the transpose of a vector or matrix, and $\mathbf{P} \in \Re^{n \times n}$ is a symmetric positive definite matrix. Then, by differentiating (11), we have,

$$\dot{V} = \frac{1}{2} (\dot{\mathbf{e}}(t)^T \mathbf{P} \mathbf{e}(t) + \mathbf{e}(t)^T \mathbf{P} \dot{\mathbf{e}}(t)). \quad (12)$$

From (10) and (12), we have,

$$\begin{aligned} \dot{V} &= \frac{1}{2} \left\{ \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t) - \mathbf{H}_m \hat{\mathbf{x}}(t) - \mathbf{B}_m \mathbf{r}(t)] \right\}^T \mathbf{P} \mathbf{e}(t) \\ &\quad + \frac{1}{2} \mathbf{e}(t)^T \mathbf{P} \sum_{i=1}^p w_i [(\mathbf{A}_i + \Delta \mathbf{A}_i) \mathbf{x}(t) + \mathbf{B} \mathbf{u}_i(t) - \mathbf{H}_m \hat{\mathbf{x}}(t) - \mathbf{B}_m \mathbf{r}(t)]. \end{aligned} \quad (13)$$

We design $\mathbf{u}_i(t)$, $i = 1, 2, \dots, p$, as follows,

$$\mathbf{u}_i = \begin{cases} \mathbf{B}^{-1} \left(\mathbf{H} \mathbf{e}(t) - \mathbf{A}_i \mathbf{x}(t) + \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t) - \frac{\mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)} \right) & \text{if } \mathbf{e}(t) \neq \mathbf{0}, \\ \mathbf{B}^{-1} \left(-\mathbf{A}_i \mathbf{x}(t) + \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t) \right) & \text{if } \mathbf{e}(t) = \mathbf{0}, \end{cases} \quad (14)$$

where $\|\cdot\|$ denotes the l_2 norm for vectors and l_2 induced norm for matrices, $\|\Delta\mathbf{A}_i\| \leq \|\Delta\mathbf{A}_i\|_{\max}$, $\mathbf{H} \in \mathfrak{R}^{n \times n}$ is a stable matrix to be designed. A block diagram of the closed-loop system is shown in Fig. 1. In (14), it is assumed that \mathbf{B}^{-1} exists (\mathbf{B} is nonsingular). In the latter part of this section, we shall provide a way to check if the assumption is valid. From (13), (14) and assuming that $\mathbf{e}(t) \neq \mathbf{0}$, we have

$$\begin{aligned}
 \dot{V} &= \frac{1}{2} \left\{ \sum_{i=1}^p w_i \left[\mathbf{H}\mathbf{e}(t) + \Delta\mathbf{A}_i\mathbf{x}(t) - \frac{\mathbf{e}(t)\|\mathbf{e}(t)\|\|\mathbf{P}\|\|\Delta\mathbf{A}_i\|_{\max}\|\mathbf{x}(t)\|}{\mathbf{e}(t)^T\mathbf{P}\mathbf{e}(t)} \right] \right\}^T \mathbf{P}\mathbf{e}(t) \\
 &\quad + \frac{1}{2} \mathbf{e}(t)^T \mathbf{P} \sum_{i=1}^p w_i \left[\mathbf{H}\mathbf{e}(t) + \Delta\mathbf{A}_i\mathbf{x}(t) - \frac{\mathbf{e}(t)\|\mathbf{e}(t)\|\|\mathbf{P}\|\|\Delta\mathbf{A}_i\|_{\max}\|\mathbf{x}(t)\|}{\mathbf{e}(t)^T\mathbf{P}\mathbf{e}(t)} \right] \\
 &= \frac{1}{2} \mathbf{e}(t)^T (\mathbf{H}^T \mathbf{P} + \mathbf{P}\mathbf{H}) \mathbf{e}(t) \\
 &\quad + \sum_{i=1}^p w_i \left(\mathbf{e}(t)^T \mathbf{P} \Delta\mathbf{A}_i \mathbf{x}(t) - \frac{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta\mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)} \right) \\
 &\leq \frac{1}{2} \mathbf{e}(t)^T (\mathbf{H}^T \mathbf{P} + \mathbf{P}\mathbf{H}) \mathbf{e}(t) \\
 &\quad + \sum_{i=1}^p w_i \left(\|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta\mathbf{A}_i\| \|\mathbf{x}(t)\| - \frac{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta\mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)} \right) \\
 &\leq -\frac{1}{2} \mathbf{e}(t)^T \mathbf{Q} \mathbf{e}(t) + \sum_{i=1}^p w_i \|\mathbf{e}(t)\| \|\mathbf{P}\| (\|\Delta\mathbf{A}_i\| - \|\Delta\mathbf{A}_i\|_{\max}) \|\mathbf{x}(t)\|, \quad (15)
 \end{aligned}$$

where $\mathbf{Q} = -(\mathbf{H}^T \mathbf{P} + \mathbf{P}\mathbf{H})$ is a symmetric positive definite matrix. As $\|\Delta\mathbf{A}_i\| - \|\Delta\mathbf{A}_i\|_{\max} \leq 0$, from (15), we have,

$$\dot{V} \leq -\frac{1}{2} \mathbf{e}(t)^T \mathbf{Q} \mathbf{e}(t) \leq 0. \quad (16)$$

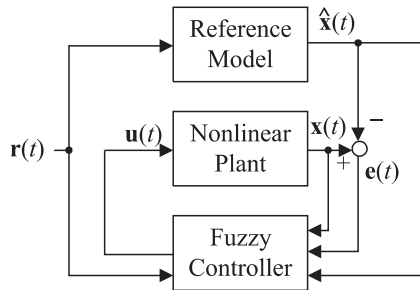


Fig. 1. A block diagram of the closed-loop system.

If $\mathbf{e}(t) = \mathbf{0}$, $\dot{V} = 0$. Hence, we can conclude that $\mathbf{e}(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$. In the following, we shall derive a sufficient condition to check the existence of \mathbf{B}^{-1} . From (9), and considering the following dynamic system,

$$\dot{\mathbf{z}}(t) = \mathbf{B}\mathbf{z}(t) = \sum_{i=1}^p w_i \mathbf{B}_i \mathbf{z}(t). \quad (17)$$

If the nonlinear system of (17) is asymptotically stable, it implies that \mathbf{B}^{-1} exists. To ensure the asymptotic stability, consider the following Lyapunov function,

$$V_z = \frac{1}{2} \mathbf{z}(t)^T \mathbf{P}_z \mathbf{z}(t), \quad (18)$$

where $\mathbf{P}_z \in \Re^{n \times n}$ is a symmetric positive definite matrix. Then, by differentiating (18), we have,

$$\dot{V}_z = \frac{1}{2} (\dot{\mathbf{z}}(t)^T \mathbf{P}_z \mathbf{z}(t) + \mathbf{z}(t)^T \mathbf{P}_z \dot{\mathbf{z}}(t)). \quad (19)$$

From (17) and (19), we have,

$$\begin{aligned} \dot{V}_z &= \frac{1}{2} \left[\left(\sum_{i=1}^p w_i \mathbf{B}_i \mathbf{z}(t) \right)^T \mathbf{P}_z \mathbf{z}(t) + \mathbf{z}(t)^T \mathbf{P}_z \sum_{i=1}^p w_i \mathbf{B}_i \mathbf{z}(t) \right] \\ &= \frac{1}{2} \sum_{i=1}^p w_i \mathbf{z}(t)^T (\mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i) \mathbf{z}(t) \\ &= -\frac{1}{2} \sum_{i=1}^p w_i \mathbf{z}(t)^T \mathbf{Q}_i \mathbf{z}(t), \end{aligned} \quad (20)$$

where $\mathbf{Q}_i = -(\mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i)$. If $\mathbf{Q}_i < \mathbf{0}$ for all $i = 1, \dots, p$, then, from (20), we have,

$$\dot{V} = -\frac{1}{2} \sum_{i=1}^p w_i \mathbf{z}(t)^T \mathbf{Q}_i \mathbf{z}(t) \leq 0. \quad (21)$$

The nonlinear system of (17) is then asymptotically stable and \mathbf{B}^{-1} exists. On the other hand, if we let $\bar{\mathbf{B}} = -\mathbf{B}$ and $\bar{\mathbf{B}}_i = -\mathbf{B}_i$, and consider $\dot{\mathbf{z}}(t) = \bar{\mathbf{B}}\mathbf{z}(t) = \sum_{i=1}^p w_i \bar{\mathbf{B}}_i \mathbf{z}(t)$. It can be shown that $\bar{\mathbf{B}}^{-1}$ exists if there exist $\mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i < \mathbf{0}$ for all $i = 1, 2, \dots, p$. The existence of $\bar{\mathbf{B}}^{-1}$ implies the existence of \mathbf{B}^{-1} . The results of this section can be summarized by the following lemma.

Lemma 1. *The fuzzy control system of (8) subject to plant parameter uncertainties is guaranteed to be asymptotically stable, and its states will follow those of a stable reference model of (1), if the following two conditions satisfy;*

(1) \mathbf{B} is nonsingular. One sufficient condition to guarantee the nonsingularity of \mathbf{B} is that there exists \mathbf{P}_z such that,

$$-(\mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i) < \mathbf{0} \quad \text{for all } i \quad \text{or} \quad \mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i < \mathbf{0} \quad \text{for all } i.$$

(2) The control laws of fuzzy controller of (7) are designed as,

$$\mathbf{u}_i = \begin{cases} \mathbf{B}^{-1} \left(\mathbf{H} \mathbf{e}(t) - \mathbf{A}_i \mathbf{x}(t) + \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t) - \frac{\mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)} \right) \\ \text{if } \mathbf{e}(t) \neq \mathbf{0}, \\ \mathbf{B}^{-1} (-\mathbf{A}_i \mathbf{x}(t) + \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t)) \quad \text{if } \mathbf{e}(t) = \mathbf{0}. \end{cases}$$

From the control laws stated in Lemma 1, it can be seen that the controller in each fuzzy controller rule can be regarded as a sliding mode controller. $\mathbf{B}^{-1}(-\mathbf{A}_i \mathbf{x}(t) + \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m \mathbf{r}(t))$ can be viewed as an equivalent control term and

$$\begin{aligned} \frac{\mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)} &\geq \frac{\mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\|\mathbf{P}\| \|\mathbf{e}(t)\|^2} \\ &= \frac{\mathbf{e}(t) \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\|\mathbf{e}(t)\|}. \end{aligned}$$

So,

$$\frac{\mathbf{e}(t) \|\mathbf{e}(t)\| \|\mathbf{P}\| \|\Delta \mathbf{A}_i\|_{\max} \|\mathbf{x}(t)\|}{\mathbf{e}(t)^T \mathbf{P} \mathbf{e}(t)}$$

can be viewed as a nonlinear (switching) term used to compensate the parameter uncertainties. Hence, we may call the proposed controller a fuzzy sliding mode controller.

The procedure for finding the fuzzy controller can be summarized as follows.

Step (I). Obtain the mathematical model of the nonlinear plant to be controlled.

Step (II). Obtain the fuzzy plant model for the system stated in step (I) by means of a fuzzy modeling method.

Step (III). Check if there exists \mathbf{B}^{-1} by finding the \mathbf{P}_z according to Lemma 1. If \mathbf{P}_z cannot be found, the design fails. \mathbf{P}_z can be found by using some existing LMI tools.

Step (IV). Choose a stable reference model.

Step (IV). Design the fuzzy controller according to Lemma 1.

4. Numerical example

A numerical example is given in this section to illustrate the procedure of finding the fuzzy controller. The fuzzy plant model of a nonlinear plant is assumed to be available, and we start from step (II) of the design procedure.

Step (II). The nonlinear plant can be represented by a fuzzy model with the following fuzzy rules,

$$\begin{aligned} \text{Rule } i : & \text{ IF } x(t) \text{ is } M_1^i \text{ AND } \dot{x}(t) \text{ is } M_2^i \\ \text{ THEN } & \dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i)\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t), \quad i = 1, 2, 3, 4, \end{aligned} \quad (22)$$

where the membership functions of M_α^i , $\alpha = 1, 2$, are shown in Figs. 2 and 3, respectively,

$$\begin{aligned} \mu_{M_1^1}(x(t)) = \mu_{M_1^2}(x(t)) &= 1 - \frac{x(t)^2}{2.25}, & \mu_{M_1^3}(x(t)) = \mu_{M_1^4}(x(t)) &= \frac{x(t)^2}{2.25}, \\ \mu_{M_2^1}(\dot{x}(t)) = \mu_{M_2^2}(\dot{x}(t)) &= 1 - \frac{\dot{x}(t)^2}{6.75}, & \mu_{M_2^3}(\dot{x}(t)) = \mu_{M_2^4}(\dot{x}(t)) &= \frac{\dot{x}(t)^2}{6.75}, \end{aligned} \quad (23)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad \mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix},$$

$$\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix}, \quad \mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 & 1 \\ -1.4387 & -2 \end{bmatrix},$$

$$\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 & 1 \\ -0.5613 & -2 \end{bmatrix},$$

$$\Delta\mathbf{A}_1 = \Delta\mathbf{A}_2 = \Delta\mathbf{A}_3 = \Delta\mathbf{A}_4 = \begin{bmatrix} 0 & 0 \\ d_1(t) & d_2(t) \end{bmatrix},$$

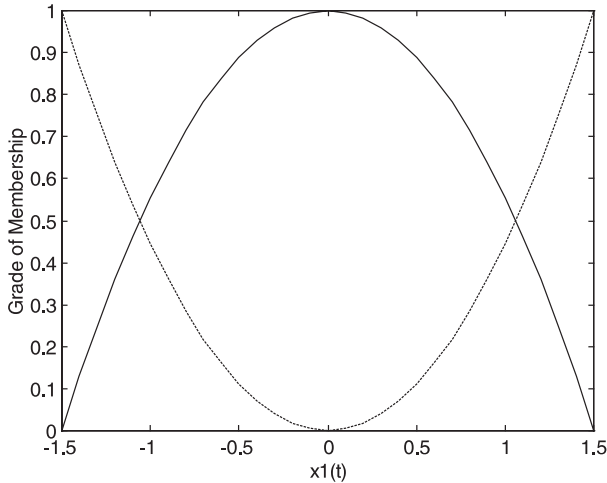


Fig. 2. Membership functions of the fuzzy plant model: $\mu_{M_1^1}(x) = \mu_{M_1^2}(x) = 1 - (x^2/2.25)$ (solid line), $\mu_{M_1^3}(x) = \mu_{M_1^4}(x) = (x^2/2.25)$ (dotted line).

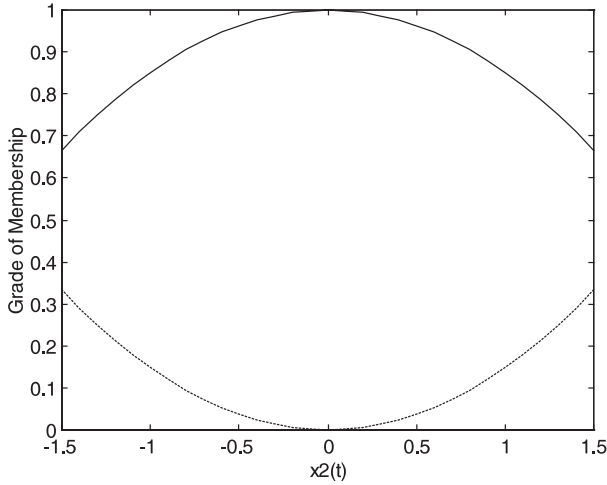


Fig. 3. Membership functions of the fuzzy plant model: $\mu_{M_1^1}(\dot{x}) = \mu_{M_1^2}(\dot{x}) = 1 - (\dot{x}^2/6.75)$ (solid line), $\mu_{M_2^1}(\dot{x}) = \mu_{M_2^2}(\dot{x}) = \dot{x}^2/6.75$ (dotted line).

where $d_1(t)$ and $d_2(t)$ are the parameter uncertainties. Practically, they are unknown values within given bounds. In this example, they are defined as time-varying functions to illustrate the robustness of the controller.

$$d_1(t) = \frac{d_1^U + d_1^L}{2} + \left(c_1^L - \frac{d_1^U + d_1^L}{2} \right) \cos(t) \text{ so that } d_1(t) \in [d_1^L, d_1^U], \quad (24)$$

$$d_2(t) = \frac{d_2^U + d_2^L}{2} + \left(c_2^L - \frac{d_2^U + d_2^L}{2} \right) \cos(t) \text{ so that } d_2(t) \in [d_2^L, d_2^U], \quad (25)$$

$$d_1^L = -0.5, \quad d_1^U = 0.5, \quad d_2^L = -0.1, \quad d_2^U = 0.1.$$

Step (III). We choose

$$\mathbf{P}_z = \begin{bmatrix} 39.7945 & 12.6915 \\ 12.6915 & 14.9997 \end{bmatrix}$$

such that

$$\mathbf{Q}_i = -(\mathbf{B}_i^T \mathbf{P}_z + \mathbf{P}_z \mathbf{B}_i) < \mathbf{0} \quad \text{for } i = 1, 2, \dots, 4.$$

Hence, we can guarantee the existing of \mathbf{B}^{-1} .

Step (IV). The stable reference model is chosen as follows,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{H}_m \hat{\mathbf{x}}(t) + \mathbf{B}_m r(t), \quad (26)$$

where

$$\mathbf{H}_m = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{B}_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (27)$$

Step (V). The rules of the fuzzy controller are designed as follows,

Rule j : IF $x(t)$ is M_1^j AND $\dot{x}(t)$ is M_2^j THEN

$$\mathbf{u}(t) = \mathbf{u}_j, \quad j = 1, 2, 3, 4, \quad (28)$$

where

$$\mathbf{u}_j = \begin{cases} \mathbf{B}^{-1} \left(\mathbf{H}\mathbf{e}(t) - \mathbf{A}_i\mathbf{x}(t) + \mathbf{H}_m\hat{\mathbf{x}}(t) + \mathbf{B}_m\mathbf{r}(t) - \frac{\mathbf{e}(t)\|\mathbf{e}(t)\|\|\mathbf{P}\|\|\Delta\mathbf{A}\|_{\max}\|\mathbf{x}(t)\|}{\mathbf{e}(t)^T\mathbf{P}\mathbf{e}(t)} \right) & \text{if } \mathbf{e}(t) \neq \mathbf{0} \\ \mathbf{B}^{-1}(-\mathbf{A}_i\mathbf{x}(t) + \mathbf{H}_m\hat{\mathbf{x}}(t) + \mathbf{B}_m\mathbf{r}(t)) & \text{if } \mathbf{e}(t) = \mathbf{0} \end{cases}$$

for $j = 1, 2, 3, 4$. We choose

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

which is a stable matrix,

$$\mathbf{P} = \begin{bmatrix} 1.5000 & 0.5000 \\ 0.5000 & 1.000 \end{bmatrix}$$

and $\|\Delta\mathbf{A}\|_{\max} = 0.5099$ (from (24) and (25)).

Figs. 4 and 5 show the system responses of the fuzzy control system without (solid lines) and with (dash lines) parameter uncertainties, and the reference model (dotted lines) under $r(t) = 0$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$. Figs. 6 and 7 show the system responses of the fuzzy control system without (solid lines) and with (dash lines) parameter uncertainties, and the reference model

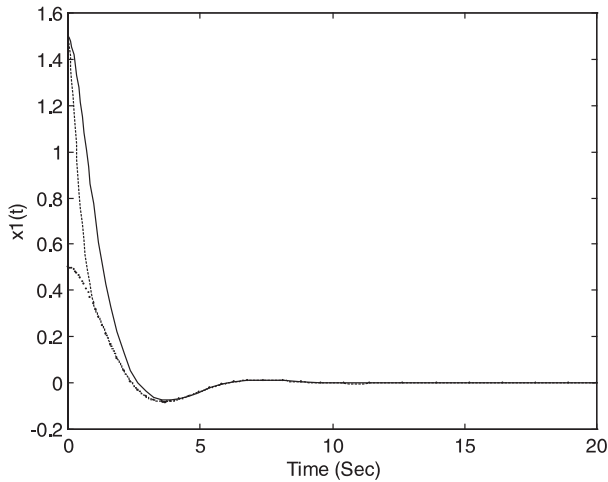


Fig. 4. Responses of $x_1(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = 0$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

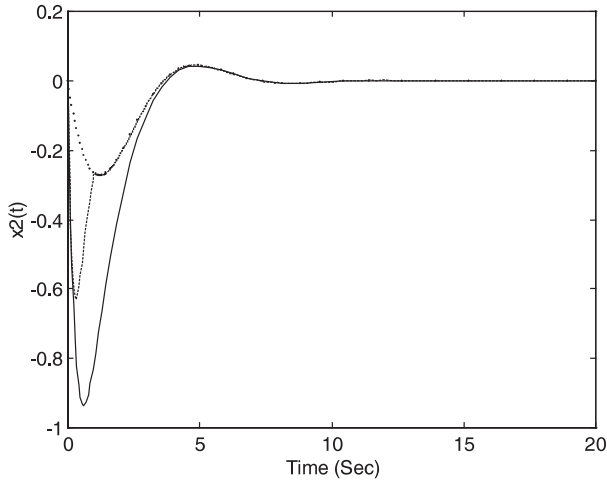


Fig. 5. Responses of $x_2(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = 0$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

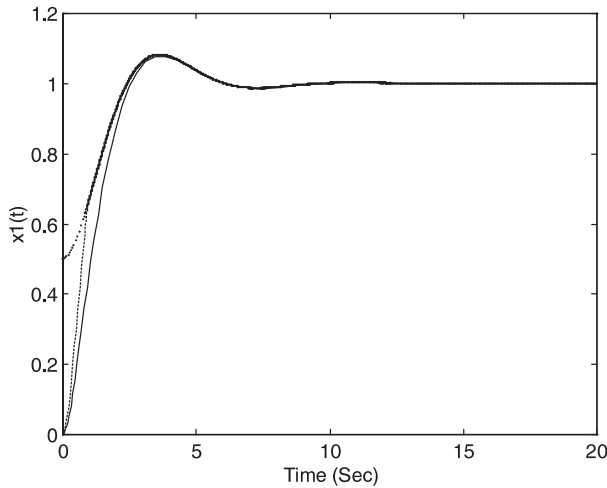


Fig. 6. Responses of $x_1(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = 1$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

(dotted lines) under $r(t) = 1$, $\mathbf{x}(0) = [0 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$. Figs. 8 and 9 show the system responses of the fuzzy control system without (solid lines) and with (dash lines) parameter uncertainties, and the reference model (dotted

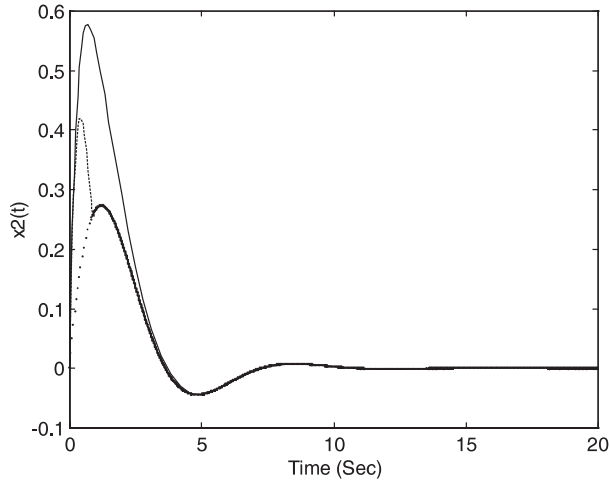


Fig. 7. Responses of $x_2(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = 1$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

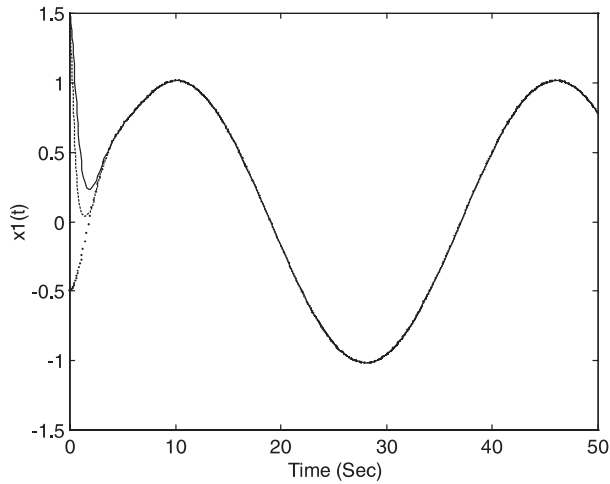


Fig. 8. Responses of $x_1(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = \sin(10t)$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

lines) under $r(t) = \sin(10t)$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [-0.5 \ 0]^T$. It can be seen from the simulation results that the system states of the nonlinear system follow those of the reference model. The responses of the fuzzy control system

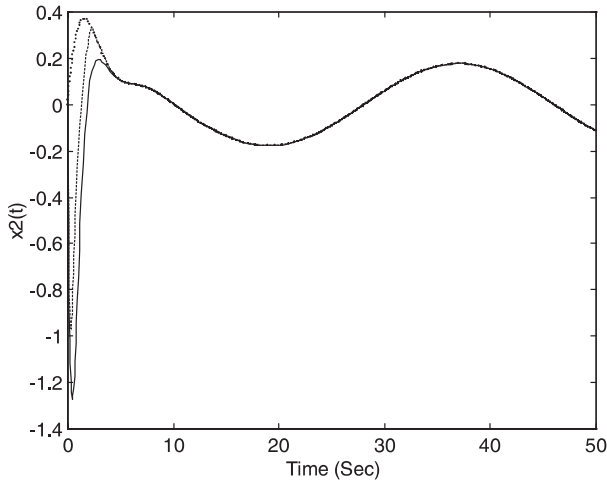


Fig. 9. Responses of $x_2(t)$ of the fuzzy control system without (solid line) and with parameter uncertainties (dash line), and the reference model (dotted line) under $r(t) = \sin(10t)$, $\mathbf{x}(0) = [1.5 \ 0]^T$ and $\hat{\mathbf{x}}(0) = [0.5 \ 0]^T$.

with parameter uncertainties are better than that of the fuzzy control system without parameter uncertainties. This is because an additional control signal, i.e.,

$$\frac{\mathbf{e}(t)\|\mathbf{e}(t)\|\|\mathbf{P}\|\|\Delta\mathbf{A}\|_{\max}\|\mathbf{x}(t)\|}{\mathbf{e}(t)^T\mathbf{P}\mathbf{e}(t)}$$

is used. The reason can also be seen from (15), i.e.,

$$\dot{V} \leq -\frac{1}{2}\mathbf{e}(t)^T\mathbf{Q}\mathbf{e}(t) + \frac{1}{2}\sum_{i=1}^P w_i\|\mathbf{e}(t)\|\|\mathbf{P}\|(\|\Delta\mathbf{A}\| - \|\Delta\mathbf{A}\|_{\max})\|\mathbf{x}(t)\|$$

the term

$$\frac{1}{2}\sum_{i=1}^P w_i\|\mathbf{e}(t)\|\|\mathbf{P}\|(\|\Delta\mathbf{A}\| - \|\Delta\mathbf{A}\|_{\max})\|\mathbf{x}(t)\|$$

makes $\mathbf{e}(t)$ approach zero at a faster rate.

5. Conclusion

Model reference fuzzy control of a class of n th-order n -input nonlinear systems subject to parameter uncertainties has been discussed. A design procedure of the fuzzy controller has been presented. The closed-loop system will

behave like a user-defined reference model. A numerical example has been given to show the design procedure and the merits of the designed fuzzy controller.

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